Stability preserving post-processing methods applied in the Loewner framework

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Outline

1 Introduction and Motivation
2 Loewner Framework
3 Post-Processing Methods
4 Numerical Examples
Model order reduction (MOR) is used to transform large, complex models of time dependent processes into smaller, simpler models that are still capable of representing accurately the behavior of the original process under a variety of conditions.

Consider interpolatory model reduction methods: we seek reduced models whose transfer function matches that of the original system at selected interpolation points.

Reduce dimension from $n = \dim(\Sigma)$ to $k = \dim(\hat{\Sigma})$

\[
\Sigma : \begin{cases}
  E\dot{x}(t) = Ax(t) + Bu(t) \\
y(t) = Cx(t)
\end{cases}
\Rightarrow
\hat{\Sigma} : \begin{cases}
  \hat{E}\hat{x}(t) = \hat{A}\hat{x}(t) + \hat{B}u(t) \\
y(t) = \hat{C}\hat{x}(t)
\end{cases}
\]
Main tool: Loewner framework $\rightarrow$ data driven MOR method.

- Uses measured or computed data (e.g. measurements of the frequency response of a to-be approximated system) instead of the system matrices ($E, A, B$ and $C$).

- Constructs reduced models based on a rank revealing factorization of appropriately constructed matrices.
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Constructs reduced models based on a rank revealing factorization of appropriately constructed matrices.

In some cases the resulting reduced systems may not be stable!

Very good approximation but unstable model → find a stable model which is an optimal or sub-optimal approximant of the original reduced Loewner model (in the $\mathcal{H}_\infty$ norm).
Loewner Framework

- Measured data: \((\omega_k, S_k)\) where \(k = \{1, \ldots, N\}\), \(\omega_k \in \mathbb{C}, S_k \in \mathbb{C}^{p \times m}\).
- Partition the data:
  \[
  \begin{cases}
  \text{right data: } (\lambda_i, r_i, w_i), & i = \{1, \ldots, \rho\} \\
  \text{left data: } (\mu_j, l_j, v_j), & j = \{1, \ldots, \nu\}
  \end{cases}
  \]

- **Objective**: Find \(H(s) \in \mathbb{C}^{p \times m}\) s.t. \(H(\lambda_i)r_i = w_i, \quad l_j^*H(\mu_j) = v_j^*\)
Loewner Framework

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  \]

The **Loewner matrix** \( L \in \mathbb{C}^{\nu \times \rho} \):

\[
L = \begin{pmatrix}
\frac{v_1 r_1 - l_1 w_1}{\mu_1 - \lambda_1} & \cdots & \frac{v_1 r_1 - l_1 w_\rho}{\mu_1 - \lambda_\rho} \\
\vdots & \ddots & \vdots \\
\frac{v_\nu r_1 - l_\nu w_1}{\mu_\nu - \lambda_1} & \cdots & \frac{v_\nu r_1 - l_\nu w_\rho}{\mu_\nu - \lambda_\rho}
\end{pmatrix}
\]

\[
V = \begin{pmatrix}
v_1^* \\
v_2^* \\
\vdots \\
v_\nu^*
\end{pmatrix}^*
\]

The shifted Loewner matrix \( L_\sigma \in \mathbb{C}^{\nu \times \rho} \):

\[
L_\sigma = \begin{pmatrix}
\frac{\mu_1 v_1 r_1 - \lambda_1 l_1 w_1}{\mu_1 - \lambda_1} & \cdots & \frac{\mu_1 v_1 r_1 - \lambda_\rho l_1 w_\rho}{\mu_1 - \lambda_\rho} \\
\vdots & \ddots & \vdots \\
\frac{\mu_\nu v_\nu r_1 - \lambda_1 l_\nu w_1}{\mu_\nu - \lambda_1} & \cdots & \frac{\mu_\nu v_\nu r_1 - \lambda_\rho l_\nu w_\rho}{\mu_\nu - \lambda_\rho}
\end{pmatrix}
\]

\[
W = \begin{pmatrix}
w_1 \\
w_2 \\
\vdots \\
w_\rho
\end{pmatrix}
\]
Theorem

If \((\mathbb{L}_\sigma, \mathbb{L})\) is regular, then \(\{\hat{E} = -\mathbb{L}, \hat{A} = -\mathbb{L}_\sigma, \hat{B} = V, \hat{C} = W\}\) is a realization of the data. Hence, \(H(z) = W(\mathbb{L}_\sigma - z\mathbb{L})^{-1}V\) is the required interpolant.
Theorem

If \((L_\sigma, L)\) is regular, then \(\{\hat{E} = -L, \hat{A} = -L_\sigma, \hat{B} = V, \hat{C} = W\}\) is a realization of the data. Hence, \(H(z) = W(L_\sigma - zL)^{-1}V\) is the required interpolant.

Minimal amount of data:

\[
\begin{pmatrix}
E & A \\
B & C
\end{pmatrix} \rightarrow \begin{pmatrix}
-L & -L_\sigma \\
V & W
\end{pmatrix}
\]

\(H(s) = W(L_\sigma - sL)^{-1}V\)

Redundant data:

\[
\begin{pmatrix}
E & A \\
B & C
\end{pmatrix} \rightarrow \begin{pmatrix}
-Y^*LX & -Y^*L_\sigma X \\
Y^*V & WX
\end{pmatrix}
\]

\(\hat{H}(s) = WX[Y^*(L_\sigma - sL)X]^{-1}Y^*V\)

- Truncate at \(k = \text{rank}(L)\) \(\Rightarrow\) SVD : \(\begin{bmatrix} L & L_\sigma \end{bmatrix} = Y\Sigma_l\hat{X}^*, \begin{bmatrix} L \\ L_\sigma \end{bmatrix} = \tilde{Y}\Sigma_rX^*\)

Theorem

The quadruple \(\{\hat{E} = -Y^*LX, \hat{A} = -Y^*L_\sigma X, \hat{B} = Y^*V, \hat{C} = WX\}\), is the realization of an approximate data interpolant.
Reduction to order 2:

\[ \mathbb{L} = \begin{bmatrix} \frac{H(\mu_1) - H(\lambda_1)}{\mu_1 - \lambda_1} & \frac{H(\mu_1) - H(\lambda_2)}{\mu_1 - \lambda_2} \\ \frac{H(\mu_2) - H(\lambda_1)}{\mu_2 - \lambda_1} & \frac{H(\mu_2) - H(\lambda_2)}{\mu_2 - \lambda_2} \end{bmatrix} \]

\[ \mathbb{L}_\sigma = \begin{bmatrix} \frac{\mu_1 H(\mu_1) - \lambda_1 H(\lambda_1)}{\mu_2 - \lambda_1} & \frac{\mu_1 H(\mu_1) - \lambda_2 H(\lambda_2)}{\mu_2 - \lambda_2} \\ \frac{\mu_1 H(\mu_2) - \lambda_1 H(\lambda_1)}{\mu_2 - \lambda_1} & \frac{\mu_1 H(\mu_2) - \lambda_2 H(\lambda_2)}{\mu_2 - \lambda_2} \end{bmatrix} \]

\[ V = \begin{bmatrix} H(\mu_1) \\ H(\mu_2) \end{bmatrix} \]

\[ W = \begin{bmatrix} H(\lambda_1) & H(\lambda_2) \end{bmatrix} \]

Interpolate 2k moments of the initial linear system (k=2):

\[ H(\mu_1), \ H(\mu_2), \ H(\lambda_1), \ H(\lambda_2) \]
Consider the following classes of linear systems

\[ \Sigma^0_{n,p,m} = \{(E, A, B, C, D) \in \Sigma_{n,p,m} | \rho(E, A) \cap i\mathbb{R} = \emptyset\} \]

\[ \Sigma^+_{n,p,m} = \{(E, A, B, C, D) \in \Sigma_{n,p,m} | \rho(E, A) \subset \mathbb{C}_+\} \]

\[ \Sigma^-_{n,p,m} = \{(E, A, B, C, D) \in \Sigma_{n,p,m} | \rho(E, A) \subset \mathbb{C}_-\} \]

\( \Sigma^+_{n,p,m} \) and \( \Sigma^-_{n,p,m} \) are the sets of the stable and antistable systems.

Let \( \Sigma \in \Sigma^0_{n,p,m} \). Find two systems so that \( \Sigma \sim (\Sigma_+ \oplus \Sigma_-) \) where:

\( \Sigma_+ = (E_+, A_+, B_+, C_+, D) \in \Sigma^+_{n_+,p,m}, \Sigma_- = (E_-, A_-, B_-, C_-, 0) \in \Sigma^-_{n_-,p,m} \).

Define the space of all rational matrix transfer functions with bounded \( \mathcal{H}_\infty \) norm (where \( \|G\|_\infty = \sup_{\omega \in \mathbb{R}} \|G(i\omega)\|_2 \)):

\[ RH_\infty = \{G \in \mathbb{R}(s)^{p \times m}, i\mathbb{R} \subset \mathcal{D}(G), \|G\|_\infty < \infty\} \]
Post-Processing Methods

Definition (The \((AP_\infty)\) Problem)

Given an unstable system \(\Sigma \in \Sigma_{\bar{n},p,m}^0\) with transfer function \(G\), find a stable system \(\hat{\Sigma} \in \bigcup_{\bar{n} \in \mathbb{N}} \Sigma_{\bar{n},p,m}^-\) whose transfer function is the best approximation of \(G\) in the space \(RH_\infty\).

\[
\|G(\Sigma) - G(\hat{\Sigma})\|_\infty = \inf_{\tilde{\Sigma} \in \bigcup_{\bar{n} \in \mathbb{N}} \Sigma_{\bar{n},p,m}^-} \|G_\Sigma - G_{\tilde{\Sigma}}\|_\infty
\]

- solving \((AP_\infty)\) for an unstable system \(\Sigma\) is equivalent to solving \((AP_\infty)\) for its antistable part \(\Sigma_+\).
- For standard systems \(\Sigma\), the reversed is also known as an *Nehari Problem*. 
Theorem (Main Result)

The unstable reduced order Loewner model is decomposed:

\[
\Sigma_L = \Sigma_- \oplus \Sigma_+ \in \Sigma^0_{k,p,m}
\]

\[
\{ E_-, A_-, B_-, C_- \} \oplus \{ E_+, A_+, B_+, C_+ \}
\]

Let the infinite gramians \( P_+ \) and \( Q_+ \) and \( \sigma_1 \) the largest Hankel singular value (i.e. \( \sigma_1 = \max (\sqrt{\text{eig}(P_+ Q_+)}). \) Take \( \gamma \geq \sigma_1 \).

\[
R_+ = Q_+ E_+ P_+ E_+^T - \gamma^2 I, \quad \hat{E}_+ = E_+^T R_+, \quad \hat{B}_+ = E_+ Q_+ B_+
\]

\[
\hat{C}_+ = C_+ P_+ E_+^T, \quad \hat{A}_+ = -A_+^T R_+ - C_+^T \hat{C}_+
\]

It follows that:

\[
\hat{\Sigma}_L = \hat{\Sigma}_- \oplus \hat{\Sigma}_+ \in \Sigma^-_{k,p,m}
\]

\[
\{ E_-, A_-, B_-, C_- \} \oplus \{ \hat{E}_+, \hat{A}_+, \hat{B}_+, \hat{C}_+ \}
\]

and the inequality holds:

\[
\sigma_1 \leq \| \Sigma_L - \hat{\Sigma}_L \|_\infty \leq \min(\gamma, \| \Sigma_+ \|_\infty)
\]
\( \Sigma_- \) = stable part of \( \Sigma_L \).

\( \Sigma_{\text{opt}} \) = optimal stable approx. of \( \Sigma_L \) in \( RH_\infty \).

\( \Sigma_{\text{sopt}}^{\gamma} \) = sub-optimal stable approx. of \( \Sigma_L \) in \( RH_\infty \) (for \( \gamma > \sigma_1 \)).

\( \Sigma_{\text{flp}} \) = flipping the unstable poles approx. of \( \Sigma_L \).
\[ \Sigma_- = \text{stable part of } \Sigma_L. \]
\[ \Sigma_{opt} = \text{optimal stable approx. of } \Sigma_L \text{ in } RH_\infty. \]
\[ \Sigma_{sopt} = \text{sub-optimal stable approx. of } \Sigma_L \text{ in } RH_\infty \text{ (for } \gamma > \sigma_1). \]
\[ \Sigma_{flp} = \text{flipping the unstable poles approx. of } \Sigma_L. \]

- The Loewner model transfer function is given \((\sigma_1 = \sigma_2 = \frac{1}{6})\)

\[ H_L(s) = \frac{1}{s+3} + \frac{2}{s-2} - \frac{1}{s-1} \]

- \[ H_- (s) = \frac{1}{s+3}, \quad \|\Sigma_L - \Sigma_-\|_{H_\infty} = \frac{1}{3} \]
- \[ H_{opt}(s) = \frac{1}{s+3} - \frac{1}{6}, \quad \|\Sigma_L - \Sigma_{opt}\|_{H_\infty} = \frac{1}{6} \]
- \[ H_{flp}(s) = \frac{1}{s+3} + \frac{2}{s+2} - \frac{1}{s+1}, \quad \|\Sigma_L - \Sigma_{flp}\|_{H_\infty} = \frac{2}{3} \]
- \[ H_{sopt}(s) = \frac{1}{s+3} - \frac{s}{6(8s^2+25s+16)}, \quad \|\Sigma_L - \Sigma_{sopt}\|_{H_\infty} = \frac{49}{150}, \quad \gamma = \frac{7}{6} \]
Scattering parameters from EM simulation (Three coupled lines)

- Data acquired by courtesy of Prof. Dr. Grivet-Talocia, Politecnico di Torino (presentation held at MORML 16, Sttutgart ’Causality Verification for Data-Driven Macromodeling’) - 401 given sample pairs \( \{ (\omega_k, S_k) \} \)
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![Singular values of the Loewner matrix](image1)

![Poles of the Loewner reduced model](image2)

![Original data vs Loewner model](image3)
Scattering parameters from EM simulation (Three coupled lines)

- Use truncation order $k = 14$. In this case, the Loewner model has 6 unstable poles (4 real and 2 complex conjugates).
Scattering parameters from EM simulation (Three coupled lines)

<table>
<thead>
<tr>
<th>$|\Sigma_L - \Sigma_{opt}|_\infty$</th>
<th>$|\Sigma_L - \Sigma_{\gamma^{opt}}|_\infty$</th>
<th>$|\Sigma_L - \Sigma_{-}|_\infty$</th>
<th>$|\Sigma_L - \Sigma_{flp}|_\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0055</td>
<td>0.0097</td>
<td>0.0369</td>
<td>0.0934</td>
</tr>
</tbody>
</table>

- Notice that as $\omega \to \infty$, the norm of the optimal error system stays constant while the norm of the sub-optimal error system $\to 0$. 
Scattering parameters from EM simulation (Three coupled lines)

- Vary the offset $\gamma - \sigma_1$ within $(10^{-5}, 10^1)$;

$$H_\infty \text{ norm of the error systems}$$

$$\| \Sigma - \Sigma_{opt} \|_{H_\infty}$$

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$$\| \Sigma - \Sigma_{opt} \|_{H_\infty}$$

$$\lim_{\gamma \to \sigma_1} \| \Sigma_L - \Sigma_{opt}^\gamma \|_\infty = \| \Sigma_L - \Sigma_{opt} \|_\infty = \sigma_1$$

$$\lim_{\gamma \to \infty} \| \Sigma_L - \Sigma_{opt}^\gamma \|_\infty = \| \Sigma_L - \Sigma_- \|_\infty = \| \Sigma_+ \|_\infty$$
For the \( II^{nd} \) experiment, consider a stable descriptor power system of dimension \( n = 7135 \) with two inputs and two outputs, i.e. \( m = p = 2 \).

By analyzing the decay of the singular values, choose for instance reduction order \( k = 82 \).

The reduced Loewner model has 9 unstable poles.
$7135^{th}$ order power system [FRM08]

Number of unstable poles for different reduction orders

Magnitude vs. Frequency

$\sigma_1(G(i\omega))$

$\sigma_2(G(i\omega))$

$\sigma_3(G(i\omega))$

$\sigma_4(G(i\omega))$
7135\textsuperscript{th} order power system [FRM08]

Thank you for your attention!
Any questions?